

# Outline of a Generalization and a Reinterpretation of Quantum Mechanics Recovering Objectivity

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## Abstract

The ESR model has been recently proposed in several papers to offer a possible solution of the problems raising from the nonobjectivity of physical properties in quantum mechanics (QM) (mainly the *objectification problem* of the quantum theory of measurement). This solution is obtained by embodying the mathematical formalism of QM into a broader mathematical framework and reinterpreting quantum probabilities as conditional on detection rather than absolute. We provide a new and more general formulation of the ESR model and discuss time evolution according to it, pointing out in particular that both linear and nonlinear evolution may occur, depending on the physical environment.

**Keywords:** quantum mechanics; ESR model; quantum measurements; evolution equations.

## 1 Introduction

It is well known that the standard interpretation of quantum mechanics (QM), though empirically successful, is a source of problems and paradoxes. One can avoid these difficulties by adopting a purely statistical interpretation of QM [1], but at the expense of accepting that QM has nothing to say about single items of physical systems (briefly, *individual objects*). If one maintains instead that QM refers to individual objects and their properties,<sup>1</sup> as we will do in the following,

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<sup>1</sup>This position is called “realistic” by some authors [2]. It expresses, however, a very weak form of realism, which does not assume any *a priori* model for individual objects and their properties and does not imply ontological commitments about the theoretical entities of QM. Moreover, one could interpret individual objects as activations of preparation procedures [3], thus avoiding any realistic interpretation at a microscopic level.

then the *objectification problem* arises which makes it difficult to work out a consistent quantum theory of measurement [2]. The deep root of this problem is *nonobjectivity* of physical properties, which intuitively means that there are in QM physical properties that may be brought into existence by a measurement but do not preexist to it (*ibid.*; see also [4]).

Nonobjectivity is strongly supported by several theorems (often dubbed “no-go” theorem), as Bell’s [5], which establishes that QM is a *nonlocal* theory, and Bell–Kochen–Specker’s [6, 7], which establishes that QM is a *contextual* theory. Both these theorems imply indeed that the outcome of the measurement of a physical property  $E$  on an individual object  $\alpha$  may depend in QM not only on  $E$  and  $\alpha$ , but also on the measurement context, even if the measurement that is performed is assumed to be *exact* (efficiency 1, no flaws or random errors in the measuring apparatus). Further support to nonobjectivity is then provided by the results of experiments, as Aspect’s [8, 9] and similar successive experiments (see [10] for a broad bibliography on this topic) which are usually interpreted as showing the nonlocality of QM.

Nonobjectivity, however, has many puzzling consequences besides the objectification problem. For instance, it entails that the usual *epistemic* notion of probability cannot be maintained in the case of quantum probabilities, which are necessarily *nonepistemic* (or *ontic*). This implies in particular that some ambiguities occur in the interpretation of mixed states (or *mixtures*) in QM.<sup>2</sup> Moreover, nonobjectivity implies, according to many scholars, that a nonclassical logic (*quantum logic*) must be adopted in the language of QM, formalizing the properties of a notion of *quantum truth* different from classical truth [14]. Furthermore, nonobjectivity is counterintuitive, as witnessed by the long-standing debate about wave–particle duality. Indeed, it entails that no intuitive model for individual objects and their relationships can be constructed because such a model would imply that the physical properties of an individual object are independent of the measurement context.

Notwithstanding the problems outlined above, all early attempts at providing a hidden variables theory for QM, as Bohm’s [15], or operational foundations of QM, as the quantum logical or the  $C^*$ -algebra approach (see, *e.g.*, [2] for a short illustration of these approaches and related biography) preserved, more or less explicitly, the contextuality and the nonlocality of QM. Also some recent efforts of recovering the structure of QM from general principles and notions inspired by quantum information theory, as Zeilinger’s foundational principle [16], CBH theorem [17], quantum Bayesianism [18, 19, 20], etc., either do not accept the weak realistic position introduced above or do not question contextuality and nonlocal correlations, which are instead considered as basic features and resources for quantum information processing. Also these approaches are strongly supported by the theorems and experimental results mentioned above.

Philosophers of science, however, know that the interpretation of experimen-

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<sup>2</sup>Indeed, all mixtures are represented by density operators in QM, but in the decomposition of these operators in terms of pure states different interpretations of the coefficients occur: epistemic probabilities, in the case of *proper* mixtures, nonepistemic probabilities, in the case of *improper mixtures* [11, 12, 13].

tal data may be different in different theories. Moreover, one of us, together with various collaborators, has shown in some previous papers that the proofs of the “no-go” theorems rest on an implicit assumption on the range of validity of physical laws that is problematic in QM [21, 22]. If this assumption is weakened, the proofs of the theorems cannot be completed. This suggests that “objective” interpretations of the formalism of QM cannot be *a priori* excluded, even if they may imply some restrictions on the range of validity of this theory. A *semantic realism* (SR) interpretation of this kind was proposed indeed in several papers [21, 23, 24, 25, 26] in which objectivity of physical properties was recovered at a semantic level, avoiding ontological commitments. More recently, two of us have proposed an *extended semantic realism* (ESR) model which modifies and generalizes the SR interpretation embodying the mathematical apparatus of QM into a broader mathematical setting that admits an objective interpretation [13, 27, 28, 29, 30, 31, 32, 33, 34]. In this model quantum probabilities are reinterpreted as *conditional on detection* rather than absolute, which provides the crucial tool for restoring objectivity without giving up the mathematical formalism of standard Hilbert space QM.

The ESR model, however, was presented in the papers quoted above mixing together a *microscopic* (purely theoretical) and a *macroscopic* (empirically interpreted) part. Moreover, time evolution was not considered in this new theoretical framework. We therefore provide in the present paper a new presentation of the ESR model in which the macroscopic part is built up as a generalization and reinterpretation of QM (Sect. 3), while the microscopic part is added as a noncontextual hidden variables theory intended to show that the ESR model admits an objective interpretation (Sect. 4). We then complete the ESR model by supplying a general treatment of time evolution within the theoretical perspective introduced by it (Sect. 5). Our presentation is preceded by a short summary of QM, intended to facilitate the comparison with the generalization and reinterpretation of QM introduced by the ESR model (Sect. 2).

To close, let us briefly comment on the advantages and limits of our proposal. It is apparent that the objectification problem and the paradoxes ensuing from it are avoided in the ESR model because of objectivity. An epistemic interpretation of quantum probabilities becomes possible, and no ambiguity occurs in the interpretation of mixtures because proper and improper mixtures have different mathematical representations. Furthermore, the ESR model supports a reinterpretation of standard quantum logic which makes it compatible with classical logic [35]. The price of these achievements is a more complicate mathematical representation of physical entities. On the experimental ground, the predictions of the ESR model can be very close to the predictions of QM if some parameters (*detection probabilities*) are sufficiently close to 1, but in some cases the predictions of the two theories may considerably differ (*e.g.*, whenever proper mixtures are considered [13]). One can then contrive experiments to check which theory is correct. There are experiments, however (as Aspect’s), which actually test conditional on detection probabilities according to the ESR model. In this case the model predicts just the results that have been obtained, thus providing an explanation of them which avoids nonlocality [34, 36].

## 2 Recalls of standard QM

We provide in this section a presentation of the basic notions of QM that will be used in the rest of this paper, with the aim of making the generalization and reinterpretation of QM in Sect. 3 as immediate and transparent as possible.

### 2.1 Fundamental physical entities

A *physical system*  $\Omega$  can be theoretically described in QM by a triple  $(\mathcal{S}, \mathcal{O}, p)$ , with  $\mathcal{S}$  a set of *states*,  $\mathcal{O}$  a set of *observables* and  $p$  a *probability mapping*.

The set  $\mathcal{S}$  is partitioned into a set  $\mathcal{P}$  of *pure* states and a set  $\mathcal{S} \setminus \mathcal{P}$  of *mixed* states, or *mixtures* and, according to some authors,  $\mathcal{S} \setminus \mathcal{P}$  must be further partitioned into a set  $\mathcal{M}$  of *proper* mixtures and a set  $\mathcal{N}$  of *improper* mixtures [11].

Coming to observables, let us denote by  $\mathbb{B}(\mathbb{R})$  the set of all Borel sets of the real line  $\mathbb{R}$ , and for every set  $\Gamma$  let  $\mathcal{P}(\Gamma)$  denote the power set of  $\Gamma$ . Then, every observable  $A \in \mathcal{O}$  is associated with a set  $\Xi_A \in \mathbb{B}(\mathbb{R})$  of *possible values* of  $A$  and a set  $\mathcal{E}_A = \{E = (A, \Sigma) : \Sigma \in \mathcal{P}(\Xi_A) \cap \mathbb{B}(\mathbb{R})\}$  of *quantum properties*. Hence the set  $\mathcal{E} = \cup_{A \in \mathcal{O}} \mathcal{E}_A$  is called the *set of all quantum properties of  $\Omega$* .

Finally, the mapping  $p$  maps  $\mathcal{S} \times \mathcal{E}$  into the interval  $[0, 1]$  of  $\mathbb{R}$ , and is such that, for every  $S \in \mathcal{S}$  and  $A \in \mathcal{O}$ , the mapping which maps  $\Sigma \in \mathcal{P}(\Xi_A) \cap \mathbb{B}(\mathbb{R})$  into  $p(S, (A, \Sigma)) \in [0, 1]$  is a probability measure on  $\Xi_A$ . Hence, for every  $S \in \mathcal{S}$  and  $E = (A, \Sigma) \in \mathcal{E}$ ,  $p(S, E)$  is called the *probability of  $E$  in  $S$* .<sup>3</sup>

### 2.2 Empirical interpretation

The theoretical entities introduced in Sect. 2.1 can be empirically interpreted on macroscopic physical entities according to the following scheme [37, 3].<sup>4</sup>

A physical system  $\Omega$  is associated with a triple  $(\Pi, \mathcal{R}, \nu)$ , with  $\Pi$  a set of *preparation procedures*,  $\mathcal{R}$  a set of *dichotomic registering devices*, whose outcomes are +1 (or *yes*) and -1 (or *no*), and  $\nu$  a mapping of  $\Pi \times \mathcal{R}$  into  $[0, 1]$ . For every  $(\pi, r) \in \Pi \times \mathcal{R}$ ,  $\nu(\pi, r)$  is the large number limit of the frequency of the outcome +1 of  $r$  whenever  $r$  is used to perform a series of registrations, each occurring (immediately) after an activation of  $\pi$ .

The mapping  $\nu$  induces two equivalence relations  $\equiv$  and  $\approx$  on  $\Pi$  and  $\mathcal{R}$ , respectively, as follows.

Let  $\pi_1, \pi_2 \in \Pi$ . Then,  $\pi_1 \equiv \pi_2$  iff for every  $r \in \mathcal{R}$ ,  $\nu(\pi_1, r) = \nu(\pi_2, r)$ .

Let  $r_1, r_2 \in \mathcal{R}$ . Then,  $r_1 \approx r_2$  iff for every  $\pi \in \Pi$ ,  $\nu(\pi, r_1) = \nu(\pi, r_2)$ .

<sup>3</sup>If one puts  $\mathcal{N} = \emptyset$ , the above scheme could refer to classical and statistical mechanics as well. Of course, for every  $S \in \mathcal{P}$  and  $E \in \mathcal{E}$ ,  $p(S, E) \in \{0, 1\}$  in classical mechanics. Furthermore,  $p(S, E)$  admits an epistemic interpretation in these theories, at variance with QM (Sect. 1).

<sup>4</sup>According to a well known epistemological perspective (*received viewpoint* [38, 39] assigning an empirical interpretation of the theoretical entities implies establishing *correspondence rules* connecting the *theoretical* language of a physical theory with its *observational* language. We do not deepen this philosophical issue here, but stress that, generally, not all theoretical entities of a theory may have a direct empirical interpretation.

Every state  $S \in \mathcal{S}$  is empirically interpreted on an equivalence class  $[\pi]_{\equiv} \in \Pi/\equiv$ , and every quantum property  $E = (A, \Sigma) \in \mathcal{E}$  on an equivalence class  $[r]_{\approx} \in \mathcal{R}/\approx$ . Measuring  $E$  in  $S$  thus means applying a registering device in  $[r]_{\approx}$  after activating a preparation procedure in  $[\pi]_{\equiv}$ , obtaining one of the outcomes *yes* and *no*.

Finally, the probability mapping  $p$  is empirically interpreted on the mapping  $\tilde{\nu}$  canonically induced by  $\nu$  on  $\Pi/\equiv \times \mathcal{R}/\approx$ . More explicitly, for every  $S \in \mathcal{S}$  and  $E \in \mathcal{E}$  corresponding to  $[\pi]_{\equiv} \in \Pi/\equiv$  and  $[r]_{\approx} \in \mathcal{R}/\approx$ , respectively,  $p(S, E) \rightarrow \tilde{\nu}([\pi]_{\equiv}, [r]_{\approx}) = \nu(\pi, r)$ .

The above empirical interpretation is sufficient for our aims in this paper. It is easy to see, however, that it can be extended to observables. In this case quantum properties can be seen as special examples of observables: to be precise, dichotomic observables.<sup>5</sup>

By considering explicitly individual objects, the empirical interpretation above can be further extended. Indeed, every activation of a preparation procedure  $\pi$  can then be assumed to prepare an individual object. Hence, when studying a physical system  $\Omega$ , one can introduce the set  $\mathcal{U}$  of all individual objects (*i.e.*, the set of all items of  $\Omega$  that have been prepared) and, for every state  $S$ , the set  $\text{ext}S \subset \mathcal{U}$  of all individual objects prepared by activating preparation procedures in the equivalence class corresponding to  $S$ . Hence one can say that an individual object  $\alpha \in \mathcal{U}$  is in the state  $S$  iff  $\alpha \in \text{ext}S$ . The family  $\{\text{ext}S\}_{S \in \mathcal{S}}$  is then a partition of  $\mathcal{U}$ . Moreover, one can make the notion of nonobjectivity of QM introduced in Sect. 1 more precise. Indeed, let us agree that a quantum property  $E$  is *objective* for an individual object  $\alpha$  according to a theory  $\mathcal{T}$  if the sentence  $E(\alpha)$  that assigns  $E$  to  $\alpha$  has a classical truth value (*true/false*) in  $\mathcal{T}$  (*value definiteness*) which is independent of any measurement procedure that can be performed on  $\alpha$  (*noncontextuality*). QM then turns out to be nonobjective in the sense that quantum properties occur in it that are not objective for some individual objects. It is important to observe, however, that also objective quantum properties occur in QM. One can show in fact that a quantum property  $E$  is objective for every individual object  $\alpha$  in the state  $S$  (briefly,  $E$  is *objective* in the state  $S$ ) iff  $p(S, E)$  is either 1 or 0 [22], in accordance with the definition of objectivity introduced in [2]. If  $p(S, E) = 1$  one then says that  $E$  is *possessed* by  $\alpha$ , while  $E$  is *not possessed* by  $\alpha$  if  $p(S, E) = 0$ . Whenever  $0 \neq p(S, E) \neq 1$ , instead,  $E$  is nonobjective for every individual object  $\alpha$  in the state  $S$  according to the standard interpretation of QM. Hence, if a measurement of  $E$  on  $\alpha$  yields the outcome *yes* (*no*), one can say that  $\alpha$

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<sup>5</sup>It is well known that the attempt at describing the dichotomic registering devices (or, more generally, the apparatuses corresponding to observables) in QM, together with their interaction with the physical system  $\Omega$ , raises further problems besides those mentioned in Sect. 1. In particular, nonobjectivity transfers to the macroscopic level, as illustrated by famous paradoxes. We avoid such problems here by adopting the above straightforward empirical interpretation of the theoretical entities of QM on the macroscopic entities in  $\Pi$  and  $\mathcal{R}$ , as usual in elementary QM. Of course, in this presentation the question of whether QM can describe such entities and their interaction with  $\Omega$  (that is, ultimately, the question of the universality of QM [2]) remains unanswered. We come back on this issue in the framework of the ESR model (Sect. 5), where objectivity of properties implies that no objectification occurs.

*displays* (does not display)  $E$  in the measurement, but the sentence  $E(\alpha)$  has no truth value before the measurement. Hence every statement asserting that  $\alpha$  possesses (does not possess)  $E$  is meaningless in this case.<sup>6</sup>

### 2.3 Mathematical representation

We adopt in this paper the standard Hilbert space representation of the physical entities introduced in Sect. 2.1. Therefore the physical system  $\Omega$  is associated with a complex separable Hilbert space  $\mathcal{H}$  with scalar product  $\langle \cdot | \cdot \rangle$ . Then, in elementary QM each pure state  $P$  is represented (up to a phase factor) by a vector  $|\psi\rangle$  in the set  $\mathcal{V}$  of all unit vectors of  $\mathcal{H}$ . More generally, states are represented by linear, positive, trace 1 operators (density operators) on  $\mathcal{H}$ . For the sake of simplicity we consider only the case in which no superselection rule occurs, so that the correspondence between the set of all states and the convex set  $\mathcal{T}(\mathcal{H})_1^+$  of all density operators in  $\mathcal{H}$  is bijective. Pure states are then bijectively represented by the extremal elements of  $\mathcal{T}(\mathcal{H})_1^+$  (hence the pure state  $P$  is represented by the one-dimensional projection operator  $\rho_P = |\psi\rangle\langle\psi|$ ), while no distinction occurs between the mathematical representations of proper and improper mixtures. Furthermore, every observable  $A \in \mathcal{O}$  is represented by a self-adjoint operator  $\hat{A}$  whose spectrum is  $\Xi_A$ . Also the correspondence between the set of all observables and the set of all self-adjoint operators is supposed to be bijective. It follows that a quantum property  $E = (A, \Sigma)$  is represented by the (orthogonal) projection operator  $P^{\hat{A}}(\Sigma)$  (equivalently,  $P^{\hat{A}}(X)$ , with  $X$  any Borel set of  $\mathbb{R}$  such that  $X \cap \Xi_A = \Sigma$ ), with  $P^{\hat{A}}$  the spectral projection-valued (PV) measure on  $\mathbb{R}$  associated with  $\hat{A}$ . Finally, for every state  $S$  and quantum property  $E = (A, \Sigma)$ , the probability  $p(S, E)$  is supplied by the Born rule

$$p(S, E) = \text{Tr}[\rho_S P^{\hat{A}}(\Sigma)], \quad (1)$$

where  $\text{Tr}$  is the trace operation and  $\rho_S$  the density operator representing  $S$ .

Whenever measurements are considered, it is usual in QM to assume that a subset exists of exact registering devices that perform measurements satisfying the Lüders rule (*first kind, ideal* measurements). To be precise, let  $\alpha$  be an individual object in the state  $S$ . Then, an ideal first kind measurement of a quantum property  $E = (A, \Sigma)$  on  $\alpha$  which yields the outcome *yes* transforms  $S$  into a final state  $S_F$  represented by the density operator

$$\rho_{S_F} = \frac{P^{\hat{A}}(\Sigma) \rho_S P^{\hat{A}}(\Sigma)}{\text{Tr}[P^{\hat{A}}(\Sigma) \rho_S P^{\hat{A}}(\Sigma)]}. \quad (2)$$

The rule expressed by Eq. (2) is often (somewhat improperly) referred in the literature as *Lüders' postulate* (LP) [37].

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<sup>6</sup>We recall that this position is weakened by the *modal interpretations* of QM, which admit that, whenever  $0 \neq p(S, E) \neq 1$ ,  $E$  could be objective for *some* individual objects in the state  $S$ . Hence the modal interpretations of QM distinguish between *dynamical states* (that can be identified with the quantum states introduced above) and *value states* (the value state of an individual object  $\alpha$  representing, in our present terms, the set of all quantum properties that are objective for  $\alpha$ ).

### 3 The ESR model

As we have anticipated in Sect. 1, the ESR model aims to provide a generalization and reinterpretation of QM which avoids nonobjectivity. The basic intuitive idea underlying this model is that the set of physical properties possessed by an individual object  $\alpha$  (which must be specified by the model, but can be maintained to be non-void, in analogy with QM, see Sect. 2.2) may be such that  $\alpha$  has nonzero probability of remaining undetected whenever a physical property  $E$  is measured on it. This “no-detection” probability generally varies with  $E$  and it is different from 0 even if no cause of inefficiency or error occurs in the measuring device that is used to perform the measurement.<sup>7</sup> We show in the next sections that such an intuitive idea, though very simple, leads to a deep reinterpretation and enlargement of the formalism of QM.

#### 3.1 Fundamental physical entities

A physical system  $\Omega$  is theoretically described in the ESR model by a 5-ple  $(\mathcal{S}, \mathcal{O}_0, p^t, p^d, p)$ , with  $\mathcal{S}$  a set of *states*,  $\mathcal{O}_0$  a set of *generalized observables* and  $p^t, p^d, p$  *probability mappings*.

The set  $\mathcal{S}$  corresponds to the set denoted with the same symbol in QM (Sect. 2.1). Hence it is partitioned into a set  $\mathcal{P}$  of *pure states*, a set  $\mathcal{M}$  of *proper mixtures* and a set  $\mathcal{N}$  of *improper mixtures*.

Coming to observables, let us adopt the same conventions established in Sect. 2.1. Then, every generalized observable  $A_0 \in \mathcal{O}_0$  corresponds to an observable  $A \in \mathcal{O}$  of QM, and it is obtained from  $A$  by adding a *no-registration outcome*  $a_0$  to the set  $\Xi_A \in \mathbb{B}(\mathfrak{R})$  of all possible values of  $A$ . Hence  $\Xi_{A_0} = \Xi_A \cup \{a_0\}$  is the Borel set of all possible values of  $A_0$  (we assume in the following that  $a_0 \in \mathfrak{R}$ , which is not restrictive: indeed, if  $\Xi_A = \mathfrak{R}$ , one can choose a bijective Borel function  $f : \mathfrak{R} \rightarrow \Xi_{f(A)}$  such that  $\Xi_{f(A)} \subset \mathfrak{R}$ , and replace  $A$  by  $f(A)$ ). By analogy with QM, every  $A_0 \in \mathcal{O}_0$  is associated with a set  $\mathcal{F}_{A_0} = \{F = (A_0, \Sigma) : \Sigma \in \mathcal{P}(\Xi_{A_0}) \cap \mathbb{B}(\mathfrak{R})\}$  of *physical properties*, and the set  $\mathcal{F}_0 = \cup_{A_0 \in \mathcal{O}_0} \mathcal{F}_{A_0}$  is called the *set of all physical properties of  $\Omega$* . In addition, we introduce the subset  $\mathcal{F} = \{F = (A_0, \Sigma) : A_0 \in \mathcal{O}_0, \Sigma \in \mathcal{P}(\Xi_{A_0} \setminus \{a_0\}) \cap \mathbb{B}(\mathfrak{R})\} \subset \mathcal{F}_0$ . This definition implies that a bijective mapping  $g : (A_0, \Sigma) \in \mathcal{F} \rightarrow (A, \Sigma) \in \mathcal{E}$  exists which maps  $\mathcal{F}$  into the set  $\mathcal{E}$  of all quantum properties of  $\Omega$ .

Finally, the mapping  $p^t$  maps  $\mathcal{S} \times \mathcal{F}_0$  into the interval  $[0, 1] \subset \mathfrak{R}$ . The mappings  $p^d$  and  $p$  map instead  $\mathcal{S} \times \mathcal{F}$  into  $[0, 1]$ . Moreover,  $p^t$  is such that,

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<sup>7</sup>A similar idea occurs in various hidden variables models that were dubbed by Fine “prism models” [40, 41, 42, 43, 44]. These models mainly aim to show that the experimental results obtained in Aspect’s and similar experiments can be explained avoiding nonlocality, but do not constitute a general theory nor introduce explicitly the reinterpretation of quantum probabilities supplied by the ESR model. A fundamental non-detectability also occurs in Khrennikov’s prequantum models of the wave type (*prequantum classical statistical field theory*, or PCSFT) combined with detection by detectors with a threshold (*threshold signal detection model*, or TSD) [45, 46, 47]. Moreover, Khrennikov’s theory is local, as the ESR model (but it is contextual, at variance with the ESR model; this difference could be, however, more apparent than real, see [34], footnote 3).

for every  $S \in \mathcal{S}$  and  $A_0 \in \mathcal{O}_0$ , the mapping which maps  $\Sigma \in \mathcal{P}(\Xi_{A_0}) \cap \mathbb{B}(\mathfrak{R})$  into  $p^t(S, (A_0, \Sigma)) \in [0, 1]$  is a probability measure on  $\Xi_{A_0}$ . Hence, for every  $S \in \mathcal{S}$  and  $F = (A_0, \Sigma) \in \mathcal{F}_0$ ,  $p^t(S, F)$  is called the *overall probability of  $F$  in  $S$* . Furthermore,  $p$  is such that, for every  $S \in \mathcal{S}$  and  $A_0 \in \mathcal{O}_0$ , the mapping which maps  $\Sigma \in \mathcal{P}(\Xi_{A_0} \setminus \{a_0\}) \cap \mathbb{B}(\mathfrak{R})$  into  $p(S, (A_0, \Sigma)) \in [0, 1]$  is a probability measure on  $\Xi_{A_0} \setminus \{a_0\}$ , while  $p^d$  is such that, for every  $S \in \mathcal{S}$  and  $\Sigma \in \mathcal{P}(\Xi_{A_0} \setminus \{a_0\}) \cap \mathbb{B}(\mathfrak{R})$ ,

$$p^t(S, (A_0, \Sigma)) = p^d(S, (A_0, \Sigma))p(S, (A_0, \Sigma)) . \quad (3)$$

Hence, for every  $S \in \mathcal{S}$  and  $F = (A_0, \Sigma) \in \mathcal{F}$ ,  $p^d(S, F)$  and  $p(S, F)$  are called the *detection probability* and the *conditional on detection probability of  $F$  in  $S$* , respectively. All these nouns are justified by the empirical interpretation to be discussed in the next section.

### 3.2 Empirical interpretation

The theoretical entities introduced in Sect. 3.1 are empirically interpreted on macroscopic physical entities according to the following scheme.

The physical system  $\Omega$  is associated with a 5-ple  $(\Pi, \mathcal{R}_0, \nu^t, \nu^d, \nu)$ . In this 5-ple  $\Pi$  is the same set of preparation procedures that occurs in the empirical interpretation of QM (Sect. 2.2). The set  $\mathcal{R}_0$  is instead a set of exact (efficiency 1) registering devices with three possible outcomes, that we label  $+1$ ,  $0$  and  $-1$ , meaning that  $0$  is the initial position of a pointer (no-registration outcome). Then,  $\nu^t$ ,  $\nu^d$  and  $\nu$  are frequency functions which map  $\Pi \times \mathcal{R}_0$  onto  $[0, 1]$ . For every  $(\pi, r_0) \in \Pi \times \mathcal{R}_0$ ,  $\nu^t(\pi, r_0)$  is the large number limit of the frequency of the outcome  $+1$  of  $r_0$  whenever  $r_0$  is used to perform a series of registrations, each occurring after an activation of  $\pi$ ;  $\nu^d(\pi, r_0)$  is the complement to 1 of the large number limit of the frequency of the outcome  $0$  of  $r_0$  in the same series of registrations;  $\nu(\pi, r_0)$  is the large number limit of the frequency of the outcome  $+1$  of  $r_0$  whenever only registrations of the series in which the outcome  $0$  did not occur are considered.

The above definitions imply that the following equation holds

$$\nu^t(\pi, r_0) = \nu^d(\pi, r_0)\nu(\pi, r_0). \quad (4)$$

The mappings  $\nu^t$  and  $\nu^d$  induce two equivalence relations  $\equiv_0$  and  $\approx_0$  on  $\Pi$  and  $\mathcal{R}_0$ , respectively, as follows.

Let  $\pi_1, \pi_2 \in \Pi$ . Then,  $\pi_1 \equiv_0 \pi_2$  iff for every  $r_0 \in \mathcal{R}_0$ ,  $\nu^t(\pi_1, r_0) = \nu^t(\pi_2, r_0)$  and  $\nu^d(\pi_1, r_0) = \nu^d(\pi_2, r_0)$ .

Let  $r_{01}, r_{02} \in \mathcal{R}_0$ . Then,  $r_{01} \approx_0 r_{02}$  iff for every  $\pi \in \Pi$ ,  $\nu^t(\pi, r_{01}) = \nu^t(\pi, r_{02})$  and  $\nu^d(\pi, r_{01}) = \nu^d(\pi, r_{02})$ .

Every state  $S \in \mathcal{S}$  is then empirically interpreted on an equivalence class  $[\pi]_{\equiv_0} \in \Pi / \equiv_0$ , and every physical property  $F = (A_0, \Sigma) \in \mathcal{F}$  on an equivalence class  $[r_0]_{\approx_0} \in \mathcal{R}_0 / \approx_0$ . Measuring  $F$  in  $S$  then means applying a registering device in  $[r_0]_{\approx_0}$  after activating a preparation procedure in  $[\pi]_{\equiv_0}$ . If one obtains



the outcome  $+1$ , one says that the result is *yes*; if one obtains the outcome  $0$  or  $-1$ , one says that the result is *no*.

Finally, the probabilities  $p^t$ ,  $p^d$  and  $p$  are empirically interpreted on the mappings  $\tilde{\nu}^t$ ,  $\tilde{\nu}^d$  and  $\tilde{\nu}$  canonically induced on  $\Pi/\equiv_0 \times \mathcal{R}_0/\approx_0$  by  $\nu^t$ ,  $\nu^d$  and  $\nu$ , respectively. More explicitly, for every  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$  corresponding to  $[\pi]_{\equiv_0} \in \Pi/\equiv_0$  and  $[r_0]_{\approx_0} \in \mathcal{R}_0/\approx_0$ , respectively,

$$p^t(S, F) \longrightarrow \tilde{\nu}^t([\pi]_{\equiv_0}, [r_0]_{\approx_0}) = \nu^t(\pi, r_0), \quad (5)$$

$$p^d(S, F) \longrightarrow \tilde{\nu}^d([\pi]_{\equiv_0}, [r_0]_{\approx_0}) = \nu^d(\pi, r_0), \quad (6)$$

$$p(S, F) \longrightarrow \tilde{\nu}([\pi]_{\equiv_0}, [r_0]_{\approx_0}) = \nu(\pi, r_0). \quad (7)$$

We must still supply an empirical interpretation of the properties in  $\mathcal{F}_0 \setminus \mathcal{F}$ . To this end, let us observe that every  $F = (A_0, \Sigma) \in \mathcal{F}_0 \setminus \mathcal{F}$  can be associated with a physical property  $F^c = (A_0, \Xi_{A_0} \setminus \Sigma) \in \mathcal{F}$  (the *complementary property* of  $F$ ), and that the correspondence is one-to-one. Hence  $F$  is interpreted on the class  $[r_0^c]_{\approx_0}$  of dichotomic registering devices corresponding to  $F^c$ . Measuring  $F$  in  $S$  thus means applying a registering device in  $[r_0^c]_{\approx_0}$  after activating a preparation procedure in  $[\pi]_{\equiv_0}$ . If one obtains the outcome  $+1$ , one says that the result is *no*; if one obtains the outcome  $0$  or  $-1$ , one says that the result is *yes*. It follows that the large number limit of the frequency of the outcome *yes* in this kind of measurement is given by  $1 - \nu^t(\pi, r_0)$ . Therefore the probability  $p^t$  is empirically interpreted as follows:

$$p^t(S, F) \longrightarrow 1 - \tilde{\nu}^t([\pi]_{\equiv_0}, [r_0^c]_{\approx_0}) = 1 - \nu^t(\pi, r_0^c). \quad (8)$$

Let us recall that the probabilities  $p^d$  and  $p$  are not defined on  $\mathcal{S} \times (\mathcal{F}_0 \setminus \mathcal{F})$ . Hence, the empirical interpretation of states, physical properties and probabilities is now complete, which is sufficient for our aims in this paper. It could obviously be extended to observables but we do not afford this task here for the sake of brevity. We observe instead that, at variance with QM, the measurement of a physical property in the ESR model is not a special case of the measurement of a generalized observable if *yes* and *no* are considered as its possible results. Indeed, the no-registration outcome does not occur explicitly as a separate outcome in this case.

By considering explicitly individual objects, the empirical interpretation provided above can be extended, as in QM. One can introduce indeed the set  $\mathcal{U}$  of all individual objects, the set  $\text{ext}S \subset \mathcal{U}$ , the partition  $\{\text{ext}S\}_{S \in \mathcal{S}}$  and the notion of objectivity as in Sect. 2.2. However, we cannot supply a criterion of objectivity in the ESR model at this stage, as we instead did in the case of QM (Sect. 2.2). We shall see in Sect. 4 that all physical properties can be maintained to be objective in the ESR model, but for the time being we will simply assume that, for every state  $S$ , a non-void set of objective physical properties exists, by analogy with QM.

### 3.3 Basic assumptions

Bearing in mind the empirical interpretation in Sect. 3.2, let us state the fundamental assumptions of the ESR model.

**AX 1.** For every  $S \in \mathcal{S}$ ,  $F \in \mathcal{F}$ ,

$$p^t(S, F) = p^d(S, F)p(S, F). \quad (9)$$

*Physical justification.* Equation (4).

**AX 2.** For every  $S \in \mathcal{S}$ ,  $F \in \mathcal{F}_0 \setminus \mathcal{F}$ ,

$$p^t(S, F) = 1 - p^t(S, F^c). \quad (10)$$

*Physical justification.* Equation (8).

Because of assumption AX 2 we will mainly consider physical properties in  $\mathcal{F} \subset \mathcal{F}_0$  in the following.

**AX 3.** Let  $P \in \mathcal{P}$  and  $F \in \mathcal{F}$ . Then the probability  $p(P, F)$  coincides with the quantum probability  $p(P, E)$ , with  $E$  the quantum property corresponding to  $F$  via the mapping  $g$  introduced in Sect. 3.1.

*Physical justification.* Assumption AX 3 implies that the ESR model embodies the basic mathematical formalism of QM. Hence this model does not formally conflict with QM, which is a fundamental requirement if one wants to take into account the outstanding empirical success of this theory.

Assumption AX 3 deeply modifies, however, the interpretation of the mathematical formalism of QM. Indeed, consider the set  $\text{ext}P$  of all individual objects in the pure state  $P$ . According to QM, whenever an exact measurement of a physical property  $E$  is performed on an individual object  $\alpha \in \text{ext}P$ , detection always occurs and the quantum rules yield the probability that the outcome *yes* is obtained (*absolute probability*). According to the ESR model, instead, whenever a measurement of  $F = g^{-1}(E)$  is performed on an individual object  $\alpha \in \text{ext}P$ , only the individual objects in a subset  $(\text{ext}P)^d \subset \text{ext}P$  are detected, and the quantum rules yield the probability that the *yes* result is obtained in a measurement whenever  $\alpha \in (\text{ext}P)^d$  (conditional on detection probability, see Sect. 3.1).

It remains to stress that the detection probability  $p^d(S, F)$  cannot be obtained by using quantum rules. We have as yet no theory which enables us to predict it: hence it must be considered a parameter to be determined empirically. However, we have proved elsewhere that some restrictions exist on its possible values [29, 31, 32].

### 3.4 Mathematical representation

The reinterpretation of quantum probabilities introduced by assumption AX 3 has some important consequences. In particular, it entails that the mathematical formalism of QM must be extended if one wants to calculate overall

probabilities. By introducing  $p^d(S, F)$  into the mathematical formalism of QM one can then obtain the mathematical representations of states, generalized observables and physical properties that must be used in the ESR model to evaluate overall and conditional on detection probabilities, as follows.

(i) *The conditional on detection probability (pure states only).* Let  $F = (A_0, \Sigma) \in \mathcal{F}$  (hence  $a_0 \notin \Sigma$ ) and  $P \in \mathcal{P}$ . Then assumption AX 3 implies that, as far as  $p(P, F)$  is concerned,  $P$  can be represented as in QM. More explicitly,  $\Omega$  is associated with a complex separable Hilbert space  $\mathcal{H}$ ,  $P$  is represented by a unit vector  $|\psi\rangle \in \mathcal{V} \subset \mathcal{H}$  or by the one-dimensional projection operator  $\rho_P = |\psi\rangle\langle\psi|$ , and the latter representation is bijective if no superselection rules occurs (Sect. 2.3). Moreover,  $A_0$  can be represented by the self-adjoint operator  $\hat{A}$  that represents, in QM, the observable  $A \in \mathcal{O}$  from which  $A_0$  is obtained (see Sect. 3.1). Hence  $F$  can be represented by the (orthogonal) projection operator  $P^{\hat{A}}(\Sigma)$  (equivalently,  $P^{\hat{A}}(X)$  with  $X$  any Borel set of  $\mathfrak{R}$  such that  $X \cap \Xi_{A_0} = \Sigma$ ), where  $P^{\hat{A}}$  is the PV measure associated with  $\hat{A}$ . Finally, the conditional on detection probability  $p(P, F)$  can be calculated by using the standard quantum rule

$$p(P, F) = \text{Tr}[\rho_P P^{\hat{A}}(\Sigma)]. \quad (11)$$

(ii) *The overall probability (pure states only).* Bearing in mind the mathematical representations above and Eq. (9), we obtain that, for every  $P \in \mathcal{P}$  and  $F = (A_0, \Sigma) \in \mathcal{F}$ ,

$$p^t(P, F) = p^d(P, F) \text{Tr}[\rho_P \int_{\Sigma} P^{\hat{A}}(d\lambda)] = \text{Tr}[\rho_P T_{P, A_0}(\Sigma)], \quad (12)$$

with

$$T_{P, A_0}(\Sigma) = p^d(P, F) \int_{\Sigma} P^{\hat{A}}(d\lambda). \quad (13)$$

Equation (13) defines a linear, bounded, positive operator which depends not only on  $F$  but also on  $P$ . It is then natural to assume that, for every pure state  $P$  and generalized observable  $A_0 \in \mathcal{O}_0$ , a mapping  $p_{P, A_0}^d : \Xi_{A_0} \rightarrow [0, 1]$  exists such that

$$T_{P, A_0}(\Sigma) = \int_{\Sigma} p_{P, A_0}^d(\lambda) P^{\hat{A}}(d\lambda) \quad (a_0 \notin \Sigma). \quad (14)$$

Hence,

$$p^t(P, F) = \text{Tr}[\rho_P \int_{\Sigma} p_{P, A_0}^d(\lambda) P^{\hat{A}}(d\lambda)] \quad (15)$$

and

$$p^d(P, F) = \frac{\text{Tr}[\rho_P \int_{\Sigma} p_{P, A_0}^d(\lambda) P^{\hat{A}}(d\lambda)]}{\text{Tr}[\rho_P \int_{\Sigma} P^{\hat{A}}(d\lambda)]}. \quad (16)$$

Therefore, as far as  $p^t(S, F)$  is concerned, the pure state  $P$  can still be represented by  $\rho_P$ . The representation of the physical property  $F$  varies instead with  $P$ , so that  $F$  is represented by the family  $\{T_{P, A_0}(\Sigma)\}_{P \in \mathcal{P}}$ .

Let us consider now a physical property  $F = (A_0, \Sigma) \in \mathcal{F}_0 \setminus \mathcal{F}$  (hence  $a_0 \in \Sigma$ ). By using Eqs. (10), (14) and (15) we obtain

$$p^t(P, F) = 1 - \text{Tr}[\rho_P T_{P, A_0}(\Xi_{A_0} \setminus \Sigma)] = \text{Tr}[\rho_P T_{P, A_0}(\Sigma)] \quad (17)$$

with

$$T_{P, A_0}(\Sigma) = I - \int_{\Xi_{A_0} \setminus \Sigma} p_{P, A_0}^d(\lambda) P^{\hat{A}}(d\lambda) \quad (a_0 \in \Sigma). \quad (18)$$

where  $I$  is the identity operator on  $\mathcal{H}$ .

For every pure state  $P$  represented by  $\rho_P$  we can thus introduce a (commutative) positive operator valued (POV) measure

$$T_{P, A_0} : \Sigma \in \mathbb{B}(\mathfrak{R}) \mapsto T_{P, A_0}(\Sigma) \in \mathcal{B}(\mathcal{H}), \quad (19)$$

where  $\mathcal{B}(\mathcal{H})$  is the set of all bounded operators on  $\mathcal{H}$ , defined by Eqs. (14) and (18). Moreover, for every Borel set  $\Sigma \subset \Xi_{A_0}$ , the family

$$\mathcal{T}_{A_0} = \{T_{P, A_0}\}_{P \in \mathcal{P}} \quad (20)$$

allows one to calculate, via Eqs. (12) or (17), the overall probability that the outcome of a measurement of the generalized observable  $A_0$  on an individual object  $\alpha$  in the state  $P$  belongs to  $\Sigma$ . Hence we can assume that  $A_0$  is represented by the family  $\mathcal{T}_{A_0}$  as far as the overall probability is concerned.

Putting together the results in (i) and (ii) we conclude that, whenever only pure states are considered, the overall and the conditional on detection probabilities can be calculated by using the representation of pure states supplied by QM, while the mathematical representation of a physical property  $F = (A_0, \Sigma) \in \mathcal{F}$  is provided by the pair  $(P^{\hat{A}}(\Sigma), \{T_P^{\hat{A}}(\Sigma)\}_{P \in \mathcal{P}})$ . The first element of the pair coincides with the representation of  $F$  supplied by QM and must be used to calculate  $p(P, F)$ , while the second element is specific of the ESR model and must be used to calculate  $p^t(P, F)$ . Analogously, the representation of a generalized observable  $A_0 \in \mathcal{O}_0$  is provided by the pair  $(\hat{A}, \mathcal{T}_{A_0})$ , where the first element of the pair coincides with the representation supplied by QM of the observable  $A \in \mathcal{O}$  from which  $A_0$  is obtained, while the second element is specific of the ESR model.

One can now stem from the results above to discuss how overall and conditional on detection probability can be expressed in the case of mixtures. For the sake of brevity we do not discuss the details of this treatment here, and only report the results that have recently been obtained by two of us [13].

Let us begin with a preliminary remark. We have mentioned in Sect. 2.1 the distinction between proper and improper mixtures. This distinction is often ignored by physicists because the two kinds of mixtures are represented by the same mathematical entities (density operators) in QM. But several scholars have pointed out that proper and improper mixtures can be empirically distinguished [12], which implies that some physical information is lost in the mathematical representation. This is the deep reason of the problems that arise in QM when one tries to provide a physical interpretation of the coefficients that occur in

the decompositions of mixtures in terms of pure states. These problems are avoided in the ESR model, which takes into account the differences in the empirical interpretations (or *operational definitions*) of the two kinds of mixtures, supplying different mathematical representations of them.

Firstly, let us consider a proper mixture  $M \in \mathcal{M}$  of the pure states  $P_1, P_2, \dots$ , with probabilities  $p_1, p_2, \dots$ , respectively. Then,  $M$  is represented in the ESR model by a family of pairs  $\{(\rho_M(F), p^d(M, F))\}_{F \in \mathcal{F}}$ , where, for every  $F = (A_0, \Sigma) \in \mathcal{F}$ ,  $\rho_M(F)$  is a density operator given by

$$\rho_M(F) = \frac{\sum_j p_j \frac{\text{Tr}[\rho_{P_j} T_{P_j, A_0}(\Sigma)]}{\text{Tr}[\rho_{P_j} P^{\hat{A}}(\Sigma)]} \rho_{P_j}}{\sum_j p_j \frac{\text{Tr}[\rho_{P_j} T_{P_j, A_0}(\Sigma)]}{\text{Tr}[\rho_{P_j} P^{\hat{A}}(\Sigma)]}} \quad (21)$$

and  $p^d(M, F)$  is a detection probability given by

$$p^d(M, F) = \sum_j p_j p^d(P_j, F). \quad (22)$$

The conditional on detection and the overall probability are given by

$$p(M, F) = \text{Tr}[\rho_M(F) P^{\hat{A}}(\Sigma)] \quad (23)$$

and

$$p^t(M, F) = \text{Tr}[\rho_M(F) T_{M, A_0}(\Sigma)], \quad (24)$$

respectively, with

$$T_{M, A_0}(\Sigma) = p^d(M, F) P^{\hat{A}}(\Sigma). \quad (25)$$

Secondly, let us consider an improper mixture  $N \in \mathcal{N}$ . Then,  $N$  can be represented by the same density operator  $\rho_N$  that represents it in QM, and the conditional on detection probability is given by

$$p(N, F) = \text{Tr}[\rho_N P^{\hat{A}}(\Sigma)]. \quad (26)$$

Because of Eq. (26) assumption AX 3 can be extended to improper mixtures. Moreover, a linear, bounded, positive operator  $T_{N, A_0}(\Sigma)$  can be introduced as in the case of pure states, whose expression is given by Eqs. (14) and (18), with  $N$  in place of  $P$ . The overall probability is then given by

$$p^t(N, F) = \text{Tr}[\rho_N T_{N, A_0}(\Sigma)]. \quad (27)$$

Hence the set of improper mixtures can be considered as an extension of the set of pure states, and improper mixtures as *generalized pure states* [13].

Coming to physical properties, Eqs. (23)–(27) show that the mathematical representation of a physical property  $F = (A_0, \Sigma) \in \mathcal{F}$  which holds in the case of pure states can be extended to mixtures. To be precise,  $F$  is represented by the pair  $(P^{\hat{A}}(\Sigma), \{T_{S, A_0}(\Sigma)\}_{S \in \mathcal{S}})$ .

As we have seen above, the difference between the mathematical representations of proper and improper mixtures corresponds to the empirical difference between the two kinds of mixtures, which is epistemologically satisfactory and avoids the interpretative problems that arise in QM. We add that the difference between the quantum and the ESR model descriptions of proper mixtures implies that possible experiments aiming to check which of the two theories provides correct predictions can be contrived [13].

### 3.5 Idealized measurements

The representations worked out in Sect. 3.4 suggest how to modify the LP of QM (Sect. 2.3) to select a class of measurements analogous to the first kind, ideal measurements of QM. To be precise, let  $\alpha$  be an individual object in a state  $S \in \mathcal{P} \cup \mathcal{N}$  (that is,  $S$  is either a pure state or an improper mixture), represented by the density operator  $\rho_S$ . Then, we assume that, for every physical property  $F = (A_0, \Sigma) \in \mathcal{F}_0$ , a (nondestructive) *idealized* measurement exists that transforms  $S$  into the final state  $S_F$  represented by the density operator

$$\rho_{S_F} = \frac{T_{S,A_0}(\Sigma)\rho_S T_{S,A_0}(\Sigma)}{\text{Tr}[T_{S,A_0}(\Sigma)\rho_S T_{S,A_0}(\Sigma)]} \quad (28)$$

if the *yes* result is obtained. In analogy with QM we call the rule expressed by Eq. (28) *generalized Lüders postulate* (GLP) in the following.

We stress that Eq. (28) does not apply if an idealized measurement is performed on an individual object in a state  $M \in \mathcal{M}$  (proper mixture). However, the state transformation induced in this case can be deduced from Eq. (28). Its expression is rather complicate [13, 31] and we do not report it here for the sake of brevity.

Finally, let us note that we will often refer in the following to the special case of a pure state  $P$  and a discrete generalized observable  $A_0$ . Therefore let us discuss how our general formulas particularize in this specific case. If  $A_0$  is obtained from a discrete observable  $A$  of QM whose set of possible outcomes is  $\Xi = \{a_1, a_2, \dots, a_N\}$ , with  $N$  finite or infinite, the set of possible outcomes of  $A_0$  is  $\Xi_{A_0} = \{a_0, a_1, a_2, \dots, a_N\}$ . Let us denote by  $P_1^{\hat{A}}, P_2^{\hat{A}}, \dots, P_N^{\hat{A}}$  the (orthogonal) projection operators associated with  $a_1, a_2, \dots, a_N$ , respectively, by the spectral decomposition of  $\hat{A}$ . Then we get from Eqs. (14) and (18)

$$T_{P,A_0} = \begin{cases} \sum_{n|a_n \in \Sigma} p_{P,A_0}^d(a_n) P_n^{\hat{A}} & \text{if } a_0 \notin \Sigma \\ I - \sum_{n|a_n \in \Xi_{A_0} \setminus \Sigma} p_{P,A_0}^d(a_n) P_n^{\hat{A}} & \text{if } a_0 \in \Sigma \end{cases} \quad (29)$$

Let  $\Sigma = \{a_k\}$ , with  $k \in \mathbb{N}_0$ . Then Eq. (29) yields

$$T_{P,A_0}(\{a_k\}) = \begin{cases} p_{P,A_0}^d(a_k) P_k^{\hat{A}} & \text{if } k \neq 0 \\ \sum_{n=1}^N (1 - p_{P,A_0}^d(a_n)) P_n^{\hat{A}} & \text{if } k = 0 \end{cases} \quad (30)$$

If we put  $F_k = (A_0, \{a_k\})$ , Eqs. (14) and (18) yield

$$p^t(P, F_k) = \begin{cases} \text{Tr}[\rho_P p_{P, A_0}^d(a_k) P_k^{\hat{A}}] & \text{if } k \neq 0 \\ \text{Tr}[\rho_P \sum_{n=1}^N (1 - p_{P, A_0}^d(a_n)) P_n^{\hat{A}}] & \text{if } k = 0 \end{cases} \quad (31)$$

Whenever the property  $F_k$  is measured and the yes outcome is obtained, Eq. (28) yields

$$\rho_{P_{F_k}} = \begin{cases} \frac{P_k^{\hat{A}} \rho_P P_k^{\hat{A}}}{\text{Tr}[P_k^{\hat{A}} \rho_P P_k^{\hat{A}}]} & \text{if } k \neq 0 \\ \frac{\sum_{m,n=1}^N (1 - p_{P, A_0}^d(a_m)) (1 - p_{P, A_0}^d(a_n)) P_m^{\hat{A}} \rho_P P_n^{\hat{A}}}{\text{Tr}[\sum_{m,n=1}^N (1 - p_{P, A_0}^d(a_m)) (1 - p_{P, A_0}^d(a_n)) P_m^{\hat{A}} \rho_P P_n^{\hat{A}}]} & \text{if } k = 0 \end{cases} \quad (32)$$

For the sake of simplicity and intuitivity we will use sometimes the representation of pure states by means of unit vectors of  $\mathcal{H}$  in the following. We therefore observe that, if  $P \in \mathcal{P}$  is represented by the unit vector  $|\psi\rangle \in \mathcal{V}$ , the state  $P_{F_k}$  after a measurement of  $F_k$  which yields result *yes* is represented by the unit vector

$$|\psi_{F_k}\rangle = \begin{cases} \frac{P_k^{\hat{A}} |\psi\rangle}{\sqrt{\langle \psi | P_k^{\hat{A}} | \psi \rangle}} & \text{if } k \neq 0 \\ \frac{\sum_{n=1}^N (1 - p_{P, A_0}^d(a_n)) P_n^{\hat{A}} |\psi\rangle}{\sqrt{\sum_{n=1}^N (1 - p_{P, A_0}^d(a_n))^2 \|P_n^{\hat{A}} |\psi\rangle\|^2}} & \text{if } k = 0 \end{cases} \quad (33)$$

If  $k \neq 0$ , Eq. (33) reproduces the standard form of the *projection postulate* that can be found in the manuals of QM. If  $k = 0$ , it shows that the initial state can be modified by the measurement even if the individual object is not detected, though this does not occur for special classes of generalized observables [30].

## 4 A family of hidden variables models for the ESR model

We intend to supply in this section a family of noncontextual hidden variables models which show that the ESR model can be considered as an *objective* theory, at variance with QM. To this end, we add a set of theoretical *microscopic* entities to the theoretical entities (that we call *macroscopic* in the following because of the empirical interpretation in Sect. 3.2) introduced in Sect. 3.1 to describe the physical system  $\Omega$ . Intuitively, the link between the macroscopic and the microscopic entities is established by the set  $\mathcal{U}$  of all individual objects introduced in Sect. 3.2. Individual objects are supposed indeed to have microscopic properties which determine the outcomes of the measurements of (macroscopic) physical properties and the probabilities introduced in Sect. 3.1. This intuitive idea can be implemented as follows.

### 4.1 Microscopic properties and states

We assume that a physical system  $\Omega$  is characterized by a set  $\mathcal{F}_\mu$  of *microscopic properties* at a microscopic level. The elements of  $\mathcal{F}_\mu$  are the *hidden variables*

of the models. Each microscopic property  $f \in \mathcal{F}_\mu$  is a mapping  $f : \alpha \in \mathcal{U} \longrightarrow f(\alpha) \in \{0, 1\}$ . Hence, for every individual object  $\alpha \in \mathcal{U}$ , the set  $\mathcal{F}_\mu$  is partitioned in two subsets, the subset  $S_\mu = \{f \in \mathcal{F}_\mu \mid f(\alpha) = 1\}$  of microscopic properties that are *possessed* by  $\alpha$ , and the subset  $\mathcal{F}_\mu \setminus S_\mu = \{f \in \mathcal{F}_\mu \mid f(\alpha) = 0\}$  of microscopic properties that are *not possessed* by  $\alpha$  (note that the terms “possessed” and “not possessed” have no empirical interpretation at this stage: hence they do not refer to any measurement procedure). The set  $S_\mu$  is then called the *microscopic state* of  $\alpha$ , and one briefly says that  $\alpha$  *is in the microscopic state*  $S_\mu$ . Furthermore, the set of all individual objects in the microscopic state  $S_\mu$  (that is, the set of all individual objects which possess all the microscopic properties that belong to  $S_\mu$ , and only those) is called the *extension* of  $S_\mu$  and is denoted by  $\text{ext}S_\mu$ , while the set of all possible microscopic states of  $\Omega$  is denoted by  $\mathcal{S}_\mu$ . It is then apparent that the family  $\{\text{ext}S_\mu\}_{S_\mu \in \mathcal{S}_\mu}$  is a partition of  $\mathcal{U}$ .<sup>8</sup> The basic link between microscopic and macroscopic entities is now established by assuming that a bijective mapping  $\varphi : f \in \mathcal{F}_\mu \longrightarrow F \in \mathcal{F} \subset \mathcal{F}_0$  exists which makes every microscopic property correspond to a physical property of the subset  $\mathcal{F}$  introduced in Sect. 3.1. Because of this assumption one can associate each microscopic state  $S_\mu$  with a set  $\{F \in \mathcal{F} \mid \varphi^{-1}(F) \in S_\mu\}$  of physical properties or, equivalently, with a set  $\{E \in \mathcal{E} \mid \varphi^{-1}(g^{-1}(E)) \in S_\mu\}$  of quantum properties.

The result of a measurement of a (macroscopic) physical property  $F \in \mathcal{F}$  on an individual object  $\alpha$  in the state  $S \in \mathcal{S}$  is explained at a microscopic level as follows. The set of all microscopic properties possessed by  $\alpha$ , that is, the microscopic state  $S_\mu$  of  $\alpha$ , induces a probability that the registering device react or, equivalently, that  $\alpha$  be detected. Let  $f = \varphi^{-1}(F)$ . Then the measurement yields the outcome *yes* if  $\alpha$  is detected and possesses  $f$  (we say that  $\alpha$  *displays*  $F$  in this case, see Sect. 2.2), while it yields the outcome *no* if  $\alpha$  is not detected or does not possess  $f$  (we say that  $\alpha$  displays the complementary property  $F^c$  of  $F$  in this case). The result of an exact measurement of a physical property  $F \in \mathcal{F}_0 \setminus \mathcal{F}$  is then explained by considering  $F^c$  in place of  $F$  and exchanging *yes* and *no*.

The explanation above implies that, whenever  $\alpha$  is detected, it displays the physical property  $F$  iff  $f \in S_\mu$ . We are thus led to introduce the following probabilities.

$p^d(S_\mu, F)$ : the *microscopic detection probability*, that is, the probability that an individual object  $\alpha$  in the microscopic state  $S_\mu$  is detected when  $F$  is measured on it.

$p(S_\mu, F)$ : the *microscopic conditional on detection probability*, that is, the probability that an individual object  $\alpha$  in the microscopic state  $S_\mu$  display  $F$  when  $F$  is measured on it and  $\alpha$  is detected (which is either 0 or 1 since  $\alpha$  either possesses  $f = \varphi^{-1}(F)$  or not, because either  $f = \varphi^{-1}(F) \in S_\mu$  or  $f = \varphi^{-1}(F) \notin S_\mu$ ).

$p^t(S_\mu, F)$ : the *microscopic overall probability*, that is, the probability that an

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<sup>8</sup>Note that the family  $\{\text{ext}S_\mu \cap \text{ext}S\}_{S_\mu \in \mathcal{S}_\mu, S \in \mathcal{S}}$  is a further partition of  $\mathcal{U}$ , some elements of which may be void.



individual object  $\alpha$  in the microscopic state  $S_\mu$  display  $F$  when  $F$  is measured on it.

Hence, we get

$$p^t(S_\mu, F) = p^d(S_\mu, F)p(S_\mu, F) \quad (34)$$

Equation (34) is purely theoretical, because one can never directly prepare an individual object in the microscopic state  $S_\mu$ . Indeed one can only choose a device  $\pi \in \Pi$  and then prepare  $\alpha$  by means of  $\pi$ , so that  $\alpha$  is in the (macroscopic) state  $S \in \mathcal{S}$  empirically interpreted on  $[\pi]_\equiv$ . For every  $\alpha$  in the state  $S$  let us therefore introduce a further conditional probability, as follows.

$p(S_\mu|S)$ : the conditional probability that an individual object  $\alpha$  is in the microscopic state  $S_\mu$  whenever it is in the state  $S$ .

We can thus associate a subset  $\mathcal{S}_{\mu|S}$  of microscopic states with every macroscopic state  $S \in \mathcal{S}$

$$\mathcal{S}_{\mu|S} = \{S_\mu \in \mathcal{S}_\mu \mid p(S_\mu|S) \neq 0\}.^9 \quad (35)$$

The joint probability that an individual object  $\alpha$  in the state  $S$  be in the microscopic state  $S_\mu \in \mathcal{S}_{\mu|S}$ , be detected and display  $F$  when  $F$  is measured on it is then given by  $p(S_\mu|S)p^t(S_\mu, F)$ . Hence the overall probability  $p^t(S, F)$  that an individual object  $\alpha$  in the state  $S$  be detected and display  $F$  when  $F$  is measured on it is<sup>10</sup>

$$p^t(S, F) = \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S)p^t(S_\mu, F). \quad (36)$$

Moreover, the detection probability  $p^d(S, F)$  that an individual object  $\alpha$  in the state  $S$  be detected when  $F$  is measured on it is given by

$$p^d(S, F) = \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S)p^d(S_\mu, F). \quad (37)$$

Let us define now

$$p(S, F) = \frac{\sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S)p^t(S_\mu, F)}{\sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S)p^d(S_\mu, F)}. \quad (38)$$

Then, for every  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$ , we obtain

$$p^t(S, F) = p^d(S, F)p(S, F). \quad (39)$$

Equation (39) coincides with Eq. (3), hence it justifies it in terms of the hidden variables (microscopic properties) that we have introduced in our models. The crucial feature of this derivation is that no-detection is caused only by the microscopic properties possessed by  $\alpha$ . Indeed, these properties determine the

<sup>9</sup>It is then easy to show that  $\mathcal{S}_{\mu|S} = \{S_\mu \in \mathcal{S}_\mu \mid \text{ext}S_\mu \cap \text{ext}S \neq \emptyset\}$ .

<sup>10</sup>For the sake of simplicity, we consider here only the discrete case. Note that the sum can be extended to all microscopic states in  $\mathcal{S}_\mu$ , because  $p(S_\mu|S) = 0$  if  $S_\mu \notin \mathcal{S}_{\mu|S}$ .

probability  $p^d(S_\mu, F)$ , while the conditional probability  $p(S_\mu|S)$  depends only on  $S$ . Hence, Eq. (37) implies that  $p^d(S, F)$  is noncontextual, in the sense that it is determined only by the microscopic properties of the individual objects in  $\text{ext}S$ , as stated, and does not occur because of flaws or lack of efficiency of the apparatus measuring  $F$  (we stress however that  $p^d(S, F)$  depends on  $F$ : if  $F = (A_0, \Sigma)$ ,  $p^d(S, F)$ , generally, is not fixed for a given generalized observables  $A_0$  but it depends on  $\Sigma$ ).

To complete our discussion it remains to consider the measurement of a property  $F \in \mathcal{F}_0 \setminus \mathcal{F}$ . To this end let us still denote by  $p^t(S_\mu, F)$  the overall probability that an individual object  $\alpha$  in the microscopic state  $S_\mu \in \mathcal{S}_{\mu|S}$  displays  $F$  when  $F$  is measured on it. Then, putting  $F^c = (A_0, \mathcal{R} \setminus \Sigma)$  as in Sect. 3.3, we introduce the physically reasonable assumption that, for every  $S_\mu \in \mathcal{S}_{\mu|S}$ ,

$$p^t(S_\mu, F) = 1 - p^t(S_\mu, F^c), \quad (40)$$

Equation (40) yields  $p^t(S_\mu, F)$  in terms of the overall probability that  $\alpha$  display  $F^c$  when  $F^c$  is measured in place of  $F$ , which is given by Eq. (34) with  $F^c$  in place of  $F$ . Then, reasoning as in the case of Eq. (36), we get

$$\begin{aligned} p^t(S, F) &= \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) p^t(S_\mu, F) = \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) (1 - p^t(S_\mu, F^c)) \\ &= \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) - \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) p^t(S_\mu, F^c). \end{aligned} \quad (41)$$

Bearing in mind that  $p^t(S, F^c) = \sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) p^t(S_\mu, F^c)$ , because of Eq. (36), and that  $\sum_{S_\mu \in \mathcal{S}_{\mu|S}} p(S_\mu|S) = 1$ , we obtain the following equation, which holds for every  $S \in \mathcal{S}$  and  $F \in \mathcal{F}_0 \setminus \mathcal{F}$ ,

$$p^t(S, F) = 1 - p^t(S, F^c) \quad (42)$$

or, equivalently

$$p^t(S, F) = 1 - p^d(S, F^c) p(S, F^c). \quad (43)$$

Equation (42) coincides with Eq. (10). Hence, also this equation is justified in terms of the hidden variables that we have introduced in our model. Moreover, also in this case  $p^d(S, F^c)$  is noncontextual in the sense specified above.

## 4.2 Recovering objectivity in the ESR model

It is well known that quantum probability is noncontextual [4]. Hence, for every  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$  the probability  $p(S, F)$  is noncontextual because of assumption AX 3. The results obtained in Sect. 4.1 then imply that also the probability  $p^t(S, F)$  is noncontextual. This conclusion is interesting, but noncontextuality of probabilities does not imply noncontextuality of the ESR model in the sense established in Sect. 2.1. Such noncontextuality (actually, objectivity) can instead be proven as follows.

The theoretical framework introduced in Sect. 4.1 provides a family of hidden variables theories, each of which is obtained by assigning different values to the probabilities  $p^d(S_\mu, F)$  and  $p(S_\mu, F)$ . These probabilities are bounded indeed only by the condition of reproducing the values of  $p^d(S, F)$  (which is a parameter to be determined experimentally, see Sect. 3.3) and  $p(S, F)$  at a macroscopic level. We can then pick out a sub-family of *deterministic* models by assuming that, for every  $S_\mu \in \mathcal{S}_\mu$  and  $F \in \mathcal{F}$ , the detection probability  $p^d(S_\mu, F)$  is either 0 or 1 (we note that this assumption does not imply that  $p^d(S, F)$  is either 0 or 1). In this case, for every  $F \in \mathcal{F}$ , detection and non-detection are completely determined by the microscopic state  $S_\mu$  of the individual object that is measured. Since also  $p(S_\mu, F)$  is either 0 or 1, as we have seen above, the result of a measurement of  $F$  on  $\alpha$  is completely determined by  $S_\mu$ . Hence value definiteness and noncontextuality occur, that is, for every  $\alpha \in \mathcal{U}$ , the physical property  $F$  is objective (Sect. 2.1). The mathematical formalism of the ESR model therefore admits an objective interpretation. Briefly, the ESR model is an objective theory.

Objectivity can be recovered also in the more general case of *probabilistic* models, in which  $p^d(S_\mu, F)$  is not bounded to be either 0 or 1, if one introduces an additional assumption. To be precise, one must suppose that a further hidden variable  $\lambda$  defined on  $\mathcal{U}$  exists such that, for every  $\alpha \in \mathcal{U}$ ,  $\lambda(\alpha)$  together with  $S_\mu$  determine whether  $\alpha$  is detected or not (which implies that the probability  $p^d(S_\mu, F)$  is epistemic) [29]. Objectivity then follows reasoning as above, with obvious changes.

Finally, we remind that the deterministic and probabilistic models have recently been connected with some previous models in the literature [34]. It has been shown indeed that the “toy” models worked out by Szabó and Fine for providing a local explanation of the results predicted in the Greenberger–Horne–Zeilinger experiment [44] are special cases of the hidden variables models for the ESR model worked out in Sect. 4.1.

## 5 Time evolution in the ESR model

Our presentation of the ESR model in Sect. 3 did not explicitly discuss time evolution, but it implicitly introduced changes of states with time when considering idealized measurements. If we assume that our generalization and representation of QM can be applied to composite systems made up of a microscopic physical system and by a macroscopic registering device (see footnote 5), we can consider this specific case as a guide for contriving a general description of time evolution in the ESR model. We therefore provide a simple measurement scheme in the next section, describing an idealized measurement as an interaction between a microscopic and a macroscopic physical system. The obtained results constitute a basis for discussing when linear unitary evolution of the composite system may occur, stating some general assumptions on time evolution and partially justifying the hypotheses on idealized measurements introduced in Sect. 3. For the sake of intuitivity, we will firstly consider pure states and

discrete generalized observables only.

### 5.1 Time evolution induced by measurements

Let  $\Omega$  be a physical system associated with the Hilbert space  $\mathcal{H}$ , and let us consider a pure state  $P$  of  $\Omega$  represented by the unit vector  $|\psi\rangle \in \mathcal{V}$ . Let  $\Omega^M$  be an apparatus (hence a macroscopic physical system) which performs a measurement of a discrete generalized observable  $A_0$  obtained from a discrete observable  $A$  of QM. By using the symbols introduced at the end of Sect. 3.4, the possible values  $a_0, a_1, a_2, \dots, a_N$  of  $A_0$  therefore correspond to positions of a pointer of  $\Omega^M$ . Let us maintain that the ESR model applies also to macroscopic physical systems and to any composite system (thus implicitly claiming the universality of the ESR model, see footnote 5). Hence  $\Omega^M$  is associated with the Hilbert space  $\mathcal{H}^M$ , and the possible values  $a_0, a_1, a_2, \dots, a_N$  correspond to states  $S_0^M, S_1^M, S_2^M, \dots, S_N^M$  of  $\Omega^M$ , respectively. In our simplified scheme these states are assumed to be pure: hence, they are represented by unit vectors  $|a_0^M\rangle, |a_1^M\rangle, |a_2^M\rangle, \dots, |a_N^M\rangle$  of  $\mathcal{H}^M$ , respectively. We denote by  $\mathcal{G}^M$  the subspace  $\langle \{|a_0^M\rangle, |a_1^M\rangle, |a_2^M\rangle, \dots, |a_N^M\rangle\} \rangle$  of  $\mathcal{H}^M$  generated by these vectors in the following.

Let us describe an *idealized* measurement performed on a item of  $\Omega$  by  $\Omega^M$ .<sup>11</sup> We introduce a first assumption on such a measurement as follows.

**AXM 1.** The Hilbert space  $\mathcal{H}$  is homomorphic to the proper subspace  $\mathcal{L}^M$  of  $\mathcal{G}^M$  generated by the set  $\{|a_1^M\rangle, |a_2^M\rangle, \dots, |a_N^M\rangle\}$  of unit vectors of  $\mathcal{G}^M$ .

Let  $\tau$  be the homomorphism whose existence is assured by assumption AXM 1, let us consider the family  $\{\mathcal{S}_n = \tau^{-1}(|a_n^M\rangle)\}_{n=1,2,\dots,N}$  of orthogonal subspaces of  $\mathcal{H}$ , and let us put  $\dim \mathcal{S}_n = g_n$ . For every  $n$ , let us choose an orthonormal basis  $\{|a_n^\mu\rangle\}_{\mu=1,2,\dots,g_n}$  on  $\mathcal{S}_n$ . Then, the set  $\{|a_n^\mu\rangle\}_{n=1,2,\dots,N;\mu=1,2,\dots,g_n}$  is an orthonormal basis on  $\mathcal{H}$ . Moreover, we have seen in Sect. 3.4 that the mathematical representation of the generalized observable  $A_0$  is provided by the pair  $(\hat{A}, \mathcal{T}_{A_0})$ . Hence, for every  $n = 1, 2, \dots, N$ ,  $a_n$  must be an eigenvalue of  $\hat{A}$ , degenerate of order  $g_n$ . The rules for calculating overall probabilities and state transformations are then given by Eqs. (31) and (33), respectively. Let us consider now the composite physical system  $(\Omega, \Omega^M)$  obtained by putting together  $\Omega$  and  $\Omega^M$ . Then pure states and improper mixtures of  $(\Omega, \Omega^M)$  can be represented as in QM according to the ESR model. Hence,  $(\Omega, \Omega^M)$  can be associated with the Hilbert space  $\mathcal{H} \otimes \mathcal{H}^M$ , and Eq. (33) suggests introducing a second assumption on idealized measurements, as follows.

**AXM 2.** Let an item of  $(\Omega, \Omega^M)$  be in a pure state represented by the unit vector  $|\Psi_0\rangle = |\psi\rangle|a_0^M\rangle \in \mathcal{H} \otimes \mathcal{H}^M$ . Then an idealized measurement of  $A_0$

<sup>11</sup>We do not use the term *individual object* in this section to avoid confusing an item of  $\Omega$  with an item of the composite system to be introduced below.

maps  $|\Psi_0\rangle$  into a unit vector  $|\Psi\rangle$ , as follows.

$$\begin{aligned} |\Psi_0\rangle &= |\psi\rangle|a_0^M\rangle = \sum_{n=1}^N \sum_{\mu=1}^{g_n} c_n^\mu |a_n^\mu\rangle |a_0^M\rangle \\ \longrightarrow |\Psi\rangle &= \sum_{n=1}^N \sum_{\mu=1}^{g_n} \alpha_{Pn} c_n^\mu |a_n^\mu\rangle |a_n^M\rangle + \beta_{P0} |\psi_{F_0}\rangle |a_0^M\rangle. \end{aligned} \quad (44)$$

The coefficients  $\alpha_{Pn}$  and  $\beta_{P0}$  in Eq. (44) are given by

$$\begin{cases} \alpha_{Pn} = \sqrt{p_{P,A_0}^d(a_n)} e^{i\theta_{Pn}} \\ \beta_{P0} = \sqrt{p_{P,A_0}^t(a_0)} e^{i\varphi_{P0}} \end{cases}, \quad (45)$$

where  $\theta_{Pn}, \varphi_{P0}$  are arbitrary real numbers and the following equation holds

$$\sum_{n=1}^N \sum_{\mu=1}^{g_n} |\alpha_{Pn} c_n^\mu|^2 + |\beta_{P0}|^2 = \langle \Psi | \Psi \rangle = 1. \quad (46)$$

It is apparent that assumption AXM 2 modifies the standard description of the measurement process in QM [48] by introducing the vector  $|\psi_{F_0}\rangle$  which represents the final state of the item of  $\Omega$  that is measured whenever the  $a_0$  outcome is obtained.

By using Eq. (44) one can write down the density operator  $\rho_\Psi = |\Psi\rangle\langle\Psi|$  representing the state of an item of  $(\Omega, \Omega^M)$  after the measurement. Hence, one can obtain the density operator  $\tilde{\rho} = \text{Tr}_{\Omega^M} \rho_\Psi$  representing the state of an item of  $\Omega$  after the measurement (Sect. 3.4). It follows indeed from Eq. (44) that

$$\begin{aligned} \tilde{\rho} &= \sum_{n=1}^N \langle a_n^M | \Psi \rangle \langle \Psi | a_n^M \rangle \\ &= |\beta_{P0}|^2 |\psi_{F_0}\rangle \langle \psi_{F_0}| + \sum_{n=1}^N |\alpha_{Pn}|^2 \sum_{\mu,\nu=1}^{g_n} c_n^\mu (c_n^\nu)^* |a_n^\mu\rangle \langle a_n^\nu| \\ &= |\beta_{P0}|^2 |\psi_{F_0}\rangle \langle \psi_{F_0}| + \sum_{n=1}^N |\alpha_{Pn}|^2 P_n^{\hat{A}} \rho_P P_n^{\hat{A}}. \end{aligned} \quad (47)$$

It is then natural to generalize Eq. (47) to every state  $S \in \mathcal{P} \cup \mathcal{N}$ , as follows.

$$\tilde{\rho} = p_{S,A_0}^t(a_0) \rho_{S_{F_0}} + \sum_{n=1}^N p_{S,A_0}^d(a_n) P_n^{\hat{A}} \rho_S P_n^{\hat{A}}, \quad (48)$$

where  $\rho_{S_{F_0}}$  is given by Eq. (32). Eq. (48) thus provides the basic equation for the state transformation induced by a measurement in the ESR model.

Finally, we note that Eq. (47) modifies the perspective in [31]: indeed, we do not obtain a proper mixture as a final state after a nonselective measurement of a generalized observable of an item of  $\Omega$  in a pure state  $P$  but, rather, an improper mixture.

## 5.2 Linear unitary evolution

The evolution of the composite physical system  $(\Omega, \Omega^M)$  postulated by assumption AXM 2 depends on the parameters  $\alpha_{P_n}$  and  $\beta_{P_0}$  that are left undetermined in Eq. (44). We may then wonder whether these parameters can be determined in such a way that the evolution is induced by a linear unitary operator. To answer this question let us refer to the symbols introduced in Sect. 5.1 and let us denote by  $S_n^\mu$  the pure state of  $\Omega$  represented by the unit vector  $|a_n^\mu\rangle$ . Then, Eq. (44) yields

$$|a_n^\mu\rangle|a_0^M\rangle \longrightarrow \alpha_{S_n^\mu} |a_n^\mu\rangle|a_n^M\rangle + \beta_{S_n^\mu} |(a_n^\mu)_{F_0}\rangle|a_0^M\rangle. \quad (49)$$

Let us assume that  $\alpha_{S_n^\mu}$  and  $\beta_{S_n^\mu}$  are real and do not depend on  $\mu$ , consistently with their physical interpretation as square roots of detection probabilities, up to a phase factor (Eq. (45)). Then, we briefly put  $\alpha_{S_n^\mu} = \sqrt{p_{S_n^\mu, A_0}^d(a_n)} = \alpha_n$ ,  $\beta_{S_n^\mu} = \sqrt{p_{S_n^\mu, A_0}^t(a_0)} = \beta_{n0}$  and  $|(a_n^\mu)_{F_0}\rangle = |a_{n0}^\mu\rangle$  (hence,  $|\alpha_n|^2 + |\beta_{n0}|^2 = 1$ ). If we now assume that the evolution of  $(\Omega, \Omega^M)$  is induced by a linear unitary operator  $U$ , we obtain from Eq. (49)

$$\begin{aligned} |\Psi_0\rangle &= |\psi\rangle|a_0^M\rangle = \sum_{n=1}^N \sum_{\mu=1}^{g_n} c_n^\mu |a_n^\mu\rangle|a_0^M\rangle \\ &\xrightarrow{U} |\Psi\rangle = \sum_{n=1}^N \sum_{\mu=1}^{g_n} c_n^\mu (\alpha_n |a_n^\mu\rangle|a_n^M\rangle + \beta_{n0} |a_{n0}^\mu\rangle|a_0^M\rangle) \\ &= \sum_{n=1}^N \sum_{\mu=1}^{g_n} \alpha_n c_n^\mu |a_n^\mu\rangle|a_n^M\rangle + \sum_{n=1}^N \sum_{\mu=1}^{g_n} \beta_{n0} c_n^\mu |a_{n0}^\mu\rangle|a_0^M\rangle. \end{aligned} \quad (50)$$

By comparing Eq. (50) with Eq. (44) we conclude that the general evolution described by Eq. (44) is linear whenever the following conditions hold for every  $|\psi\rangle \in \mathcal{H}$

$$\begin{cases} \alpha_{P_n} = \alpha_n \\ \beta_{P_0} |\psi_{F_0}\rangle = \sum_{n=1}^N \sum_{\mu=1}^{g_n} \beta_{n0} c_n^\mu |a_{n0}^\mu\rangle \end{cases} \quad (51)$$

We can thus maintain that Eqs. (51) are satisfied and that  $(\Omega, \Omega^M)$  undergoes linear unitary evolution.<sup>12</sup> The density operator  $\tilde{\rho}$  is given in this case by

$$\tilde{\rho} = |\beta_{\psi 0}|^2 |\psi_{F_0}\rangle\langle\psi_{F_0}| + \sum_{n=1}^N |\alpha_n|^2 P_n^{\hat{A}} \rho_P P_n^{\hat{A}}, \quad (52)$$

where

$$|\beta_{P_0}| = \left| \sum_{n=1}^N \sum_{\mu=1}^{g_n} \beta_{n0} c_n^\mu |a_{n0}^\mu\rangle \right| \quad (53)$$

<sup>12</sup>The evolution described by Eq. (50) coincides with the evolution postulated in [21]. Hence the latter is a special case of the general evolution described by Eq. (44).

and

$$|\psi_{F_0}\rangle = \frac{\sum_{n=1}^N \sum_{\mu=1}^{g_n} \beta_{n0} c_n^\mu |a_{n0}^\mu\rangle}{|\beta_{P0}|}. \quad (54)$$

One may now wonder whether the linear unitary evolution of  $(\Omega, \Omega^M)$  could lead to linear unitary evolution of the subsystem  $\Omega$  in the ESR model. Eq. (52) implies that this is not the case and that the reduced dynamics induced by a measurement is necessarily nonlinear. Let us prove this statement in the case of a nondegenerate observable (the generalization is immediate). By putting  $p_n = |\alpha_n c_n|^2$ ,  $p = |\beta_{P0}|^2$  and  $|\psi_{F_0}\rangle = \sum_{n=1}^N k_n |a_n\rangle$ , we get from Eq. (52)

$$\begin{aligned} \tilde{\rho} &= p \sum_{n,n'=1}^N k_n k_{n'}^* |a_n\rangle \langle a_{n'}| + \sum_{n=1}^N p_n |a_n\rangle \langle a_n| \\ &= \sum_{n,n'=1, n \neq n'}^N p k_n k_{n'}^* |a_n\rangle \langle a_{n'}| + \sum_{n=1}^N (p_n + p |k_n|^2) |a_n\rangle \langle a_n| \end{aligned} \quad (55)$$

If linearity holds,  $\tilde{\rho}$  represents a pure state: hence it reduces to a one-dimensional projection operator  $P$  on  $\mathcal{H}$ . Let us put  $P = |\phi\rangle \langle \phi|$  and  $|\phi\rangle = \sum_{n=1}^N d_n |a_n\rangle$ . Then we get

$$P = \sum_{n,n'=1, n \neq n'}^N d_n d_{n'}^* |a_n\rangle \langle a_{n'}| + \sum_{n=1}^N d_n d_n^* |a_n\rangle \langle a_n|, \quad (56)$$

that is, coefficients  $d_1, d_2, \dots, d_N$  exist such that, for every  $n, n' = 1, 2, \dots, N$ ,  $n \neq n'$ ,

$$d_n d_{n'}^* = p k_n k_{n'}^*, \quad (57)$$

while, for every  $n = 1, 2, \dots, N$ ,

$$d_n d_n^* = p_n + p |k_n|^2. \quad (58)$$

Eqs. (57) and (58) imply that, for every  $n, n' = 1, 2, \dots, N$ ,  $n \neq n'$ , the following condition holds.

$$(p_n + p |k_n|^2)(p_{n'} + p |k_{n'}|^2) = p^2 |k_n|^2 |k_{n'}|^2. \quad (59)$$

Eq. (59) is satisfied iff, for every  $n = 1, 2, \dots, N$ ,  $p_n = 0$ , which implies that no detection occurs. Hence we conclude that the time evolution induced by a measurement procedure on an item of  $\Omega$  is necessarily nonlinear, as stated.

### 5.3 General assumptions on time evolution

Our treatment in Sects. 5.1 and 5.2 allows us to draw, if linear Hamiltonian evolution of the composite system  $(\Omega, \Omega^M)$  is postulated, the following commutative diagram

$$\begin{array}{ccc} \rho_{\Psi_0} = |\psi\rangle \langle \psi| \otimes |a_0^M\rangle \langle a_0^M| & \xrightarrow{U} & \rho_\Psi = |\Psi\rangle \langle \Psi| \\ \text{\scriptsize } Tr_{\Omega^M} \downarrow & & \downarrow \text{\scriptsize } Tr_{\Omega^M} \\ \rho_P = |\psi\rangle \langle \psi| & \longrightarrow & \tilde{\rho} \end{array} \quad (60)$$

The density operator  $\tilde{\rho}$  is not a projection operator (Sect. 5.2). To be precise,  $\tilde{\rho}$  is diagonal in a basis  $\{|b_m\rangle\}_m$  in which it takes the form  $\tilde{\rho} = \sum_m p_m |b_m\rangle\langle b_m|$ , where at least two values  $r$  and  $s$  exist such that  $p_r \neq 0 \neq p_s$ . Hence the pure state  $\rho_P$  evolves into the improper mixture represented by  $\tilde{\rho}$ , and its evolution is not linear, even if the evolution of the composite system  $(\Omega, \Omega_M)$  is linear.

To avoid contradiction preserving linear evolution as far as possible in the ESR model, one can assume the following general rules for the time evolution of a state  $S \in \mathcal{P} \cup \mathcal{N}$  represented by the density operator  $\rho_S(t)$  at time  $t$ .

(i) *Closed systems*: linear evolution ruled by the von Neumann–Liouville equation

$$i\hbar \frac{d\rho_S(t)}{dt} = [\hat{H}, \rho_S(t)], \quad (61)$$

where  $\hat{H}$  is a self-adjoint Hamiltonian.

(ii) *Open systems*: non-necessarily linear evolution, as exemplified by the mapping  $\rho_P \rightarrow \tilde{\rho}$  in the diagram above. Whenever the open system can be considered as a subsystem of a closed system, its dynamics can be deduced from the dynamics of the closed system, and it may be linear or not depending on the Hamiltonian of the closed system.

One thus converges to a standard perspective in QM [49, 50], but avoids the problems that occur in QM because of nonobjectivity, as we show in the next section.

It remains to discuss time evolution in the case of proper mixtures. Let us therefore consider a *generalized proper mixture*  $M$  of the states  $S_1, S_2, \dots \in \mathcal{P} \cup \mathcal{M}$ , with probabilities  $p_1, p_2, \dots$ , respectively. In this case, it is natural to assume that  $S_1, S_2, \dots$  change with time according to the rules supplied above, while  $p_1, p_2, \dots$  do not change. This assumption is sufficient to provide the desired evolution. Moreover, it implies that the state transformation induced on a proper mixture by an idealized measurement of a discrete generalized observable  $A_0$  can be deduced from Eq. (48). The explicit form of this transformation has been studied in some previous papers [13, 31], and we do not report it here for the sake of brevity.

## 5.4 Justifying GLP

The change of state of  $(\Omega, \Omega^M)$  postulated in Eq. (47) has been hypothesized bearing in mind the final state of  $\Omega$  specified by Eq. (32). Conversely, let us show that Eq. (47) justifies the special form of GLP specified by Eq. (32) if one resorts to the interpretation of the measurement process in terms of hidden variables provided in Sect. 4.1.

The final state after an idealized measurement of a discrete generalized observable  $A_0$  on an item  $\alpha$  of  $\Omega$  in a pure state  $P$  is represented by the density operator  $\tilde{\rho}$  in Eq. (47). This operator can be written in the form

$$\tilde{\rho} = |\beta_{P0}|^2 |\psi_{F_0}\rangle\langle\psi_{F_0}| + \sum_{n=1}^N \gamma_{Pn} \frac{P_n^{\hat{A}} \rho_P P_n^{\hat{A}}}{\text{Tr}[P_n^{\hat{A}} \rho_P P_n^{\hat{A}}]} \quad (62)$$



where  $\gamma_{P_n} = |\alpha_{P_n}|^2 \text{Tr}[P_n^{\hat{A}} \rho_P P_n^{\hat{A}}]$ . Equation (62) provides a decomposition of  $\tilde{\rho}$  in terms of (nonorthogonal) pure states. But the coefficients  $\gamma_{P_1}, \gamma_{P_2}, \dots, \gamma_{P_N}$  that occur in it do not represent probabilities of the corresponding pure states. Rather, they represent the overall probabilities that the physical properties  $F_1, F_2, \dots, F_N$ , respectively, be displayed in the measurement. These probabilities are epistemic in the ESR model, because this model is an objective theory (Sect. 4.2). Hence they formalize our *a priori* ignorance of the outcome that will be obtained in the measurement. Whenever the display of the apparatus measuring  $A_0$  is observed, this ignorance is reduced, and we are informed that  $\alpha$  displays a specific property, say  $F_k = (A_0, \{a_k\})$ . Thus we can update our information about the properties of  $\alpha$ . Let  $k \neq 0$ . Then, according to the hidden variables models in Sect. 4.1,  $\alpha$  possesses the microscopic property  $f_k = \varphi^{-1}(F_k)$ . Therefore, if the measurement is repeated, it must yield the same result with certainty if  $\alpha$  is detected. This means that the conditional on detection probability of  $F_k$  after the first measurement is 1, which is just what occurs if  $\alpha$  is in the pure state represented by  $\frac{P_k^{\hat{A}} \rho_P P_k^{\hat{A}}}{\text{Tr}[P_k^{\hat{A}} \rho_P P_k^{\hat{A}}]}$  after the measurement, as predicted by Eq. (32). Yet, no “collapse of the wave function” occurs, because  $F_k$  is not “brought into existence” by the measurement, as in QM. Let  $k = 0$ . Then, we deduce that the set of microscopic properties possessed by  $\alpha$  is such that no-detection may occur. If the measurement is repeated, either no-detection occurs again, or one of the physical properties  $F_1, F_2, \dots, F_N$  is displayed. If we assume that the conditional on detection probability of  $F_k$  is given by  $\text{Tr}[\psi_{F_0} \langle \psi_{F_0} | P_k^{\hat{A}}]$ , we can maintain that the state of  $\alpha$  after the measurement is the pure state represented by  $|\psi_{F_0}\rangle$ , as predicted by Eq. (32).

To close, let us observe that the above reasoning does not justify the general form of GLP provided by Eq. (28). This justification would require a generalization of our arguments. Indeed, consider again a discrete generalized observable  $A_0$  and an item of  $\Omega$  in the pure state  $P$ . Performing an idealized measurement of  $A_0$  is equivalent to measuring all physical properties in the set  $\mathcal{F}_{A_0} = \{(A_0, \Sigma) \mid \Sigma \subset \Xi_0\}$  associated with  $A_0$ . Each of these measurements yields as final state one of the states represented by the density operators in Eq. (32). Hence the state after it coincides with the state predicted by GLP only for physical properties in the subset  $\{(A_0, \{a_0\}) \mid a_n \in \Xi_0\} \subset \mathcal{F}_{A_0}$ . In different words, the measurement of the physical property  $F = (A_0, \Sigma) \in \mathcal{F}_{A_0}$  is an idealized measurement in the sense established by Eq. (28) only if  $\Sigma = \{a_n\}$ , with  $n = 0, 1, 2, \dots, N$ . To justify the general form of GLP one should consider idealized measurements of generalized observables obtained from  $A_0$  by grouping together sets of eigenvalues of  $A_0$  and considering each set as a single eigenvalue. This procedure is rather obvious and we do not discuss it here for the sake of brevity.

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